Stochastic mechanics of mixed states

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Abstract. Nelson's stochastic interpretation of quantum mechanics is extended from the case of pure states to that of mixed states. It is shown that a pure probabilistic formalism, which applies the Newton–Nelson Law to the initial position and velocity distributions, does not reproduce the time evolution predicted by quantum mechanics. In order to recover the latter, a new notion must be introduced, that of pure quantum states, over which the mixture has to be decomposed, and which then satisfy the Newton–Nelson Law independently.

1. Introduction

In a previous paper Nelson (1966) has shown how to recover the Schrödinger equation, which determines the quantum evolution of a particle, by describing the movement of the latter with a stochastic process of a Brownian type, and by imposing the Newton law: $F = ma$, with F the force, m the mass and a the acceleration. Such a description has several advantages.

(i) The Schrödinger equation follows naturally from the Newton law, once fluctuations around the classical trajectories of the particle are taken into account.

(ii) The quantum fluctuations of the particle position acquire an ordinary statistical nature, and can be interpreted as resulting from a universal agitation induced by the vacuum, (proportional to $1/m$).

(iii) The wavefunction $\Psi(x)$ (which describes a linear evolution of the state of the particle) is related in a natural way to the probability $\rho(x)$ of finding the particle at a given point x in space: more precisely, the relation $\rho(x) = |\Psi(x)|^2$ follows directly from the probabilistic description and the dynamics.

This stochastic representation of the quantum mechanics of a particle thus seems to provide a natural framework for dealing simultaneously with fluctuations which are of quantum origin and fluctuations which are of pure statistical origin, like those resulting from an insufficient knowledge of the observed system. This extension of the stochastic representation of quantum mechanics, from the case of pure states to that of mixed states, is studied in this paper. In § 2, we apply quantum mechanics to the density matrix of the mixture and display the equations which describe the correct time evolution of the probability density. In § 3, following Nelson, we develop the stochastic process representation and introduce the additional hypotheses which allow one to recover the correct evolution of the mixture.

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2. Evolution of the density matrix

In the framework of quantum mechanics, all the available information on a statistical mixture is contained in its density matrix $\hat\rho$ (Landau and Lifschitz 1977). Let $\hat\rho(x, x')$ be the coefficients of this matrix in a basis of eigenvectors for the position. The probability density associated with point x is given by the diagonal element, $(\hat\rho(x, x) = \rho(x))$, while the non-diagonal ones provide phase coherence and allow for interference. According to quantum mechanics, the time evolution of the density matrix is linear and determined by the Schrödinger equation, which couples the two kinds of elements

$$i\hbar\, \partial\hat\rho/\partial t = [\hat H, \hat\rho]$$

where $\hat H$ is the Hamiltonian of the system. Before coming to a stochastic interpretation, and in order to exhibit the probabilistic features of this evolution, we shall extract the equations which the diagonal elements of the density matrix satisfy. For this purpose, let us introduce the convenient parametrisation

$$\hat\rho(x, x') = \exp(r(x, x') + is(x, x')).$$

The hermiticity of $\hat\rho$ implies the symmetries

$$r(x, x') = r(x', x) \qquad s(x, x') = -s(x', x).$$

Considering the movement of a particle in an external potential $V(x)$, the Schrödinger equation gives

$$i\hbar\, \partial\hat\rho/\partial t = [-(\hbar^2/2m)\nabla^2 + V(x) + (\hbar^2/2m)\nabla'^2 - V(x')]\hat\rho$$

with $\nabla = \partial/\partial x$, $\nabla' = \partial/\partial x'$, which becomes, after decomposition on the symmetric (real) and antisymmetric (imaginary) parts

$$\frac{\partial r}{\partial t} = -\frac{\hbar}{2m}(\nabla^2 - \nabla'^2)s - \frac{\hbar}{m}(\nabla r\, \nabla s - \nabla' r\, \nabla' s)$$

$$\frac{\partial s}{\partial t} = \frac{\hbar}{2m}(\nabla^2 - \nabla'^2)r + \frac{\hbar}{2m}((\nabla r)^2 - (\nabla s)^2 - (\nabla' r)^2 + (\nabla' s)^2) - \frac{1}{\hbar}(V(x) - V(x')).$$

$$(1)$$

In order to exploit these equations on the diagonal elements of $\hat\rho$, it appears convenient to take successive derivatives, and thus to introduce an independent set of derivatives of $\hat\rho$, taken at $x = x'$. Such a basis is, for instance, given by

$$u^{(n)}(x) = (\hbar/2m)(\nabla - \nabla')^{2n} r(x, x')|_{x=x'} \qquad v^{(n)}(x) = (\hbar/2m)(\nabla - \nabla')^{2n-1} s(x, x')|_{x=x'}$$

$$(u^{(0)}(x) = (\hbar/2m) \ln\rho(x))$$

and their derivatives with respect to x

$$\nabla^{n'} u^{(n)}(x) = (\hbar/2m)(\nabla + \nabla')^{n'}(\nabla - \nabla')^{2n} r(x, x')|_{x=x'}$$

$$\nabla^{n'} v^{(n)}(x) = (\hbar/2m)(\nabla + \nabla')^{n'}(\nabla - \nabla')^{2n-1} s(x, x')|_{x=x'}.$$

One thus obtains from (1) the set of equations which determine the time evolution of

the diagonal elements of the density matrix[†]

$$\frac{\partial}{\partial t} u^{(n)} = -\frac{\hbar}{2m} \nabla v^{(n+1)} - \sum_{p=0}^{n} \left[\binom{2n}{2p} \nabla u^{(n-p)} v^{(p+1)} + \binom{2n}{2p+1} u^{(p+1)} \nabla v^{(n-p)} \right]$$

$$\frac{\partial}{\partial t} v^{(n)} = \frac{\hbar}{2m} \nabla u^{(n)} + \sum_{p=1}^{n} \binom{2n-1}{2p-1} \left(\nabla u^{(n-p)} u^{(p)} - \nabla v^{(p)} v^{(n-p+1)} \right) - \frac{\nabla^{2n-1} V}{m}.$$

(2)

For instance, introducing

$$u(x) = \frac{\hbar}{2m} \frac{\nabla \rho}{\rho}(x) = \frac{\hbar}{m} \nabla r(x, x')|_{x=x'}, \qquad v(x) = v^{(1)}(x) = \frac{\hbar}{m} \nabla s(x, x')|_{x=x'},$$

$$u'(x) = u^{(1)}(x) = \frac{\hbar}{m} (\nabla^2 - \nabla \nabla') r(x, x')|_{x=x'}$$

the first two equations can be written

$$\partial \rho / \partial t + \nabla(\rho v) = 0 \qquad \partial v / \partial t + v \nabla v - [u + (\hbar/2m)\nabla] u' = -\nabla V / m.$$

(3)

Before coming to the stochastic description, let us make a few remarks. Equations (2) represent the time evolution of the probability density $\rho(x)$, as is required by quantum mechanics. In general, the latter involves, besides the velocity field $v(x)$, an infinity of auxiliary fields $u^{(n)}(x)$ and $v^{(n)}(x)$, which correspond to higher derivatives of the density matrix. Such a system closes on a finite number of equations and fields for particular cases only. Such is the case of pure states. Indeed, $r(x, x')$ and $s(x, x')$ then take the following forms

$$r(x, x') = R(x) + R(x') \qquad s(x, x') = S(x) - S(x')$$

$$u = (\hbar/m)\nabla R \qquad v = (\hbar/m)\nabla S \qquad u' = \nabla u$$

so that the derivatives become dependent

$$(\nabla + \nabla')^{2n} r(x, x') = (\nabla - \nabla')^{2n} r(x, x') \qquad u^{(n)}(x) = (\hbar/2m)\nabla^{2n} \ln \rho(x)$$

$$(\nabla + \nabla')^{2(n-1)} (\nabla - \nabla') s(x, x') = (\nabla - \nabla')^{2n-1} s(x, x') \qquad v^{(n)}(x) = \nabla^{2(n-1)} v(x)$$

and system (2) reduces to only two equations

$$\partial \rho / \partial t + \nabla(\rho v) = 0 \qquad \partial v / \partial t + v \nabla v - [u + (\hbar/2m)]\nabla u = -\nabla V / m.$$

(3')

One then recognises in equations (3'), for pure states, the two equations which describe the Newtonian mechanics for a stochastic process (Nelson 1966), respectively the kinematic equation of continuity, and the dynamic equation which can be rewritten $a = F/m$, where a is Nelson's acceleration which one can define for the stochastic process (see the appendix for a review of the definitions).

Coming back to the general case of the statistical mixture, one also recognises in the first two equations (3), the expression of Newtonian mechanics, but this time with a corrected acceleration. Denoting

$$\delta u(x) = \nabla u(x) - u'(x) = 2(\hbar/m)\nabla \nabla' \ln \hat{\rho}(x, x')|_{x=x'}$$

[†] To be precise, the nth pair of equations requires that $\hat{\rho}$ and V be differentiable $2n+2$ and $2n-1$ times respectively. But, as will be seen in the following, the first two equations appear to be sufficient for the argument.

the dynamic equation can be rewritten

$$\partial v/\partial t + v\,\nabla v - [u + (\hbar/2m)\nabla]\nabla u + [u + (\hbar/2m)\nabla]\delta u = -\nabla V/m.$$

In general, this correction will require Nelson's scheme to be modified in consequence. To be convinced that this correction cannot be cancelled, one can just consider the simplest case of a mixture of two different states (denoted by 1 and 2). The density matrix is then the sum of the two density matrices describing the pure states

$$\hat{\rho}(x, x') = c_1\hat{\rho}_1(x, x') + c_2\hat{\rho}_2(x, x') \qquad c_1 + c_2 = 1.$$

Denoting

$$\hat{\rho}_{1,2} = \exp(r_{1,2} + is_{1,2}) \qquad \rho_{1,2}(x) = \hat{\rho}_{1,2}(x, x')|_{x=x'}$$

$$u_{1,2}(x) = (\hbar/m)\nabla r_{1,2}(x, x')|_{x=x'} \qquad v_{1,2}(x) = (\hbar/m)\nabla s_{1,2}(x, x')|_{x=x'}.$$

One easily computes

$$u = c_1\rho_1 u_1/\rho + c_2\rho_2 u_2/\rho \qquad v = c_1\rho_1 v_1/\rho + c_2\rho_2 v_2/\rho$$

$$\delta u = 2(m/\hbar)(c_1 c_2\rho_1\rho_2/\rho^2)[(u_1 - u_2)^2 + (v_1 - v_2)^2].$$

Let us also remark that, quite generally, the three fields ($u, v, \delta u$) determine the energy of the mixture

$$E = \mathrm{Tr}\,\hat{H}\hat{\rho} = \int dx\,dx'\,\delta(x - x')[(\hbar^2/2m)\nabla\nabla' + V(x))\hat{\rho}(x, x')$$

$$= \int dx\{\tfrac{1}{2}m[u^2(x) + v^2(x) + (\hbar/2m)\delta u(x)] + V(x)\}\rho(x).$$

3. Stochastic interpretation

Being hermitian, the density matrix of the mixture can always be decomposed as a weighted sum, (with coefficients C_i), of pure state density matrices, ($\hat{\rho}_i$). Then, using Nelson's representation, we shall describe the mixture by the stochastic process which one obtains by mixing with probabilities C_i, those which describe the pure states. Let us consider for simplicity a mixture of two states, which will be denoted by 1 and 2. The position of the particle is represented by a random variable x. Let $\rho_1(x)$ and $\rho_2(x)$ be the probability densities for the random variable to take the value x in states 1 and 2 respectively. In the mixture the particle can be in states 1 and 2 with probabilities C_1 and C_2 and hence the probability density $\rho(x)$ for the random variable to take the value x is given by $\rho(x) = C_1\rho_1(x) + C_2\rho_2(x)$.

$C_1\rho_1(C_2\rho_2)$ is the joint probability density of being in state 1 and at x (in 2 and at x). $C_1\rho_1/\rho(C_2\rho_2/\rho)$ is the probability density of being in state 1 (2), with the condition of being at x. These properties can be deduced from the following hypothesis.

(a) The set of sample paths associated with the mixture is the union of the sets of sample paths associated with the states 1 and 2, and the corresponding measure is the weighted sum, with coefficients C_1 and C_2, of the measures associated with states 1 and 2.

It is then easy to recover for the mixture, the mean values which define the forward and backward velocities and which one needs for developing Nelson's cinematics

(Nelson's scheme is summarised in the appendix):

$$\lim_{\Delta t \to 0_-} \frac{\langle x(t+\Delta t)\rangle_{x(t)} - x(t)}{\Delta t} = b(x) = \frac{C_1\rho_1(x)}{\rho(x)} b_1(x) + \frac{C_2\rho_2(x)}{\rho(x)} b_2(x)$$

$$\lim_{\Delta t \to 0_+} \frac{x(t) - \langle x(t-\Delta t)\rangle_{x(t)}}{\Delta t} = b^*(x) = \frac{C_1\rho_1(x)}{\rho(x)} b_1^*(x) + \frac{C_2\rho_2(x)}{\rho(x)} b_2^*(x)$$

$$\lim_{\Delta t \to 0_+} \frac{\langle [x(t \pm \Delta t) - x(t)]^2\rangle_{x(t)}}{\Delta t} = \frac{\hbar}{m}$$

where $\langle \ \rangle_{x(t)}$ is the mean value, conditionally in $x(t)$, or else

$$\rho v = \rho_1 v_1 + \rho_2 v_2 \qquad \rho u = \rho_1 u_1 + \rho_2 u_2.$$

where $v = \frac{1}{2}(b + b^*)$ and $u = \frac{1}{2}(b - b^*)$. These relations allow one to recover the two continuity equations associated with the mixture

$$\partial \rho / \partial t + \nabla(\rho v) = 0 \qquad u = (\hbar/2m)\nabla \rho / \rho.$$

Let us furthermore make the following hypothesis.

(b) Each state evolves independently and according to the Newton–Nelson law: this means that the Newton–Nelson law ($F = ma$, with a Nelson's acceleration) is statistically satisfied in each, as taken separately, of the two sets of sample paths associated with states 1 and 2

$$\frac{\partial}{\partial t} v_1 + v_1 \nabla v_1 - u_1 \nabla u_1 - \frac{\hbar}{2m} \nabla^2 u_1 = \frac{F}{m}$$

$$\frac{\partial}{\partial t} v_2 + v_2 \nabla v_2 - u_2 \nabla u_2 - \frac{\hbar}{2m} \nabla^2 u_2 = \frac{F}{m}$$

(see the appendix for a justification of this acceleration). Summing these two equations with respective weights $C_1\rho_1/\rho$ and $C_2\rho_2/\rho$, and introducing the convenient change of variables

$$v_1 = v + (C_2\rho_2/\rho)(v_1 - v_2) \qquad u_1 = u + (C_2\rho_2/\rho)(u_1 - u_2)$$

$$v_2 = v - (C_1\rho_1/\rho)(v_1 - v_2) \qquad u_2 = u - (C_1\rho_1/\rho)(u_1 - u_2)$$

one derives the following equation

$$\frac{\partial}{\partial t} v + v \nabla v - u \nabla u - \frac{\hbar}{2m} \nabla^2 u + \left(u + \frac{\hbar}{2m} \nabla \right) \delta u = \frac{F}{m}$$

where

$$\delta u = 2(m/\hbar)(C_1 C_2 \rho_1 \rho_2 / \rho^2)[(u_1 - u_2)^2 + (v_1 - v_2)^2].$$

Thus, with the hypotheses (a) and (b) on the set of sample paths, the mixture satisfies a modified Newton–Nelson law, the correction in the acceleration taking the form $[u + (\hbar/2m)\nabla]\delta u$ and giving back the time evolution of quantum mechanics (equation

(3)). Let us remark that this last Newton law implies the conservation of the following energy

$$E = \int dx \left[\frac{m}{2} \left(u^2(x) + v^2(x) + \frac{\hbar}{2m} \delta u(x) \right) + V(x) \right] \rho(x)$$

$$\frac{dE}{dt} = \int dx \left[\frac{\partial}{\partial t} v + v \, \nabla v - \left(u + \frac{\hbar}{2m} \nabla \right) (\nabla u - \delta u) - \frac{F}{m} \right] m \rho v.$$

With this stochastic interpretation of the quantum mechanics equations, a few remarks are in order. First of all, one cannot simply apply the Newton–Nelson law to the associated stochastic process and recover, in a pure probabilistic framework, the time evolution of quantum mechanics. Indeed, if one specifies at initial time, in quite a classical way, the position and velocity distributions (the fields $\rho(x)$ and $v(x)$), and then one applies to them Nelson's evolution (equations (3')), these fields do not evolve according to quantum mechanics in the general case. In fact, to recover the latter, one must impose the modified equations (2), and this simultaneously requires that an infinite number of fields are known at initial time. Or else one must know the total density matrix at initial time, which amounts to knowing which states, and with which probabilities, build the initial mixture. Thus certain states, which are called pure states, are distinguished by quantum mechanics, and have the characteristics of evolving according to Nelson's generalisation of the Newton law. The stochastic interpretation does not avoid this but, on the contrary, must be completed with the introduction of this new notion, and then with a more precise description of the initial conditions, which must contain an explicit decomposition over pure states.

5. Conclusion

When extended to the description of mixed states, the stochastic representation still shows the advantageous features, (presented in the introduction), which bring quantum mechanics closer to the geometric and spatial background of classical mechanics. In particular, it allows one to give a precise statistical meaning to the paths and to the measure, which appear in Feynman integrals (Zambrini and Yasue 1982). It also provides a physical meaning to the fluctuations of the Markov Euclidian fields, with which the quantum field theories are constructed (Guerra 1981).

However, and already at the pure state level, there are characteristic features of quantum mechanics which are preserved by the stochastic representation, and which go beyond the standard framework of classical mechanics. In particular, the transition probability which determines the evolution of the random variable of the stochastic process is not determined by the dynamics only, but depends on the state itself. This feature leads to properties which affect the usual intuition with regard to diffusion processes (Mielnik and Tengstrand 1980), and tends to confer a physical existence to the wavefunction describing the quantum state. The random variable can be considered to be guided by the wavefunction (de Broglie 1956, Bohm 1952). The time evolution of the random variable is then more easily seen to depend crucially on the boundary conditions, with their non-local character (like, for instance, the aperture or closure of one of the slits in a double slit system). Moreover, it appears impossible to separate the evolution of the random variable from that of the boundary conditions (Wootters and Zurek 1979). The non-locality and the spatial rigidity of states, which characterise

quantum mechanics, are also exhibited in a situation suggested by Einstein *et al* (1935), and confirmed by experiments (Aspect *et al* 1982). We have seen how the notion of a quantum state also shows up in the stochastic treatment of a statistical mixture, through the necessity of a global decomposition of the set of sample paths into disjoint and priviledged subsets, which separately satisfy the Newton law. Such a decomposition is again non-local and raises, in a more acute way, the problem of the geometric and statistical characterisation of the quantum trajectories.

Acknowledgments

Appendix

We briefly review the definitions which allow the derivation of the Schrödinger equation from the Newton law. One assumes that the random variable $x(t)$ follows a locally Gaussian Markov process: the particle being at x at instant t, its probability of being at $x + \Delta x$ at instant $t + \Delta t$ is given by a transition probability $P(x, t, x + \Delta x, t + \Delta t)$ satisfying

$$\langle \Delta x \rangle_{x(t)} = \int \Delta x \, P(x, t, x + \Delta x, t + \Delta t) \, \mathrm{d}\Delta x = b(x) \, \Delta t + \mathrm{o}(\Delta t)$$

$$\langle \Delta x^2 \rangle_{x(t)} = \int \Delta x^2 \, P(x, t, x + \Delta x, t + \Delta t) \, \mathrm{d}\Delta x = \frac{k}{m} \, \Delta t + \mathrm{o}(\Delta t)$$

where k/m is the diffusion constant of the process and the other moments are of higher order in Δt. The probability density $\rho(x, t)$ then evolves according to the equation

$$\rho(x + \Delta x, t + \Delta t) = \int P(x, t, x + \Delta x, t + \Delta t) \rho(x, t) \, \mathrm{d}x$$

whose development up to order Δt leads to the Focker–Planck equation (see for instance, Cox and Miller 1980)

$$\partial \rho / \partial t + \nabla(b\rho) - (k/2m)\nabla^2 \rho = 0.$$

The same process can also be viewed another way and described by its other transition probability: the particle being at x at instant t, $P^*(x, t, x - \Delta x, t - \Delta t)$ is its probability of coming from point $x - \Delta x$ at instant $t - \Delta t$. The latter is in fact determined, once the transition probability P and the probability density ρ are known. For that purpose, let us consider $p(x, t, x', t')$ the joint probability for the particle of being at x at instant t, and at x' at instant t'. This probability can be obtained in two ways

$$p(x, t, x', t') = \rho(x, t)P(x, t, x', t') = \rho(x', t')P^*(x', t', x, t)$$

which gives

$$P^*(x, t, x - \Delta x, t - \Delta t) = [\rho(x - \Delta x, t - \Delta t)/\rho(x, t)]P(x - \Delta x, t - \Delta t, x, t).$$

P^* also describes a locally Gaussian Markov process, as its moments satisfy

$$\langle \Delta x \rangle^*_{x(t)} = \int \Delta x\, P^*(x, t, x - \Delta x, t - \Delta t)\, \mathrm{d}\Delta x = b^*(x)\, \Delta t + \mathrm{o}(\Delta t)$$

$$\langle \Delta x^2 \rangle^*_{x(t)} = \int \Delta x^2\, P^*(x, t, x - \Delta x, t - \Delta t)\, \mathrm{d}\Delta x = \frac{k}{m}\, \Delta t + \mathrm{o}(\Delta t)$$

with $b^*(x) = b(x) - 2\bar{u}(x)$, $\bar{u}(x) = (k/2m)(\nabla\rho/\rho)(x)$, the other moments being of higher order in Δt. The backward Focker–Planck equation is deduced (see Cox and Miller 1980)

$$\partial\rho/\partial t + \nabla(b^*\rho) + (k/2m)\nabla^2\rho = 0$$

which, together with the first one, leads to the two kinematic equations

$$\partial\rho/\partial t + \nabla(v\rho) = 0 \qquad \nabla[\bar{u}\rho - (k/2m)\nabla\rho] = 0$$

with

$$v = \tfrac{1}{2}(b + b^*) \quad \text{and} \quad \bar{u} = \tfrac{1}{2}(b - b^*).$$

From the moments of the transition probabilities, one can deduce the evolution of any function f of the random variable x

$$\lim_{\Delta t \to 0_-} \frac{\langle f[x(t + \Delta t)] \rangle_{x(t)} - f(x(t))}{\Delta t} = \left(\frac{\partial}{\partial t} + b(x)\nabla + \frac{k}{2m}\nabla^2 \right) f(x) = Df(x)$$

$$\lim_{\Delta t \to 0_-} \frac{f(x(t)) - \langle f[x(t - \Delta t)] \rangle_{x(t)}}{\Delta t} = \left(\frac{\partial}{\partial t} + b^*(x)\nabla - \frac{k}{2m}\nabla^2 \right) f(x) = D^*f(x)$$

which define two different time derivatives, forward D, and backward D^*, or else

$$\tfrac{1}{2}(D + D^*) = \partial/\partial t + v\nabla \qquad \tfrac{1}{2}(D - D^*) = \bar{u}\nabla + (k/2m)\nabla^2.$$

These then provide two different velocities (forward $b(x) = Dx$, and backward $b^*(x) = D^*x$), and then four different accelerations: DDx, DD^*x, D^*Dx, D^*D^*x. In fact, only three of them are independent, as can be seen from the relation which the continuity equation implies (from now on v will be assumed to derive from a potential: rot $v = 0$)

$$\tfrac{1}{2}[DD - D^*D^*]x = [\tfrac{1}{2}(D + D^*)\tfrac{1}{2}(D - D^*) + \tfrac{1}{2}(D - D^*)\tfrac{1}{2}(D + D^*)]x$$

$$= \frac{\partial}{\partial t}\,\bar{u} + \nabla(\bar{u}v) + \frac{k}{2m}\nabla^2 v = \frac{k}{2m}\nabla\left(\frac{1}{\rho}\frac{\partial}{\partial t}\rho + \frac{\nabla\rho}{\rho}v + \nabla v \right) = 0.$$

The most general acceleration then takes the form

$$\{\tfrac{1}{4}(D + D^*)^2 + \tfrac{1}{2}(D - D^*)[\tfrac{1}{2}\lambda(D - D^*) + \tfrac{1}{2}\mu(D + D^*)]\}x$$

$$= \partial v/\partial t + v\,\nabla v + [\bar{u}\nabla + (k/2m)\nabla^2](\lambda\bar{u} + \mu v).$$

If one requires not to break the time reversal invariance, $(D \leftrightarrow D^*)$, which the formalism has preserved up to now, the general acceleration reduces to

$$[\tfrac{1}{4}(D + D^*)^2 + \tfrac{1}{4}\lambda(D - D^*)^2]x$$

$$= \tfrac{1}{4}(1 + \lambda)(DD + D^*D^*)x + \tfrac{1}{4}(1 - \lambda)(DD^* + D^*D)x$$

$$= \partial v/\partial t + v\,\nabla v + \lambda[\bar{u}\nabla + (k/2m)\nabla^2]\bar{u}.$$

Let us consider the general Newton law

$$[\tfrac{1}{4}(D+D^*)^2+\tfrac{1}{4}\lambda(D-D^*)^2]x = F/m \tag{4}$$

together with the kinematic equation

$$[\tfrac{1}{2}(D+D^*)\tfrac{1}{2}(D-D^*)+\tfrac{1}{2}(D-D^*)\tfrac{1}{2}(D+D^*)]x = 0.$$

This system of equations can then be transformed in the following way.

First, linear combinations of these two equations allow one to raise the coupling between the fields

$$[\tfrac{1}{2}(D+D^*)\pm\tfrac{1}{2}\sqrt{\lambda}(D-D^*)]^2 x = F/m$$

$$\Leftrightarrow[(\partial/\partial t)+(v\pm\sqrt{\lambda}\,\bar{u})\nabla\pm\sqrt{\lambda}(k/2m)\nabla^2](v\pm\sqrt{\lambda}\,\bar{u}) = F/m.$$

Then, introducing the following change of functions

$$\Psi^\pm(x) = \exp[R(x)\pm(|\lambda|/\lambda)^{1/2}S(x)] \qquad \text{with } (k/m)\nabla R = \bar{u}, \sqrt{|\lambda|}(k/m)\nabla S = v$$

the two equations linearise

$$\pm\sqrt{\lambda}\,k(\partial/\partial t)\Psi^\pm+\lambda(k^2/2m)\nabla^2\Psi^\pm = -V\Psi^\pm \qquad \text{with } -\nabla V = F.$$

One then remarks that the probability $\rho(x)$ density is directly linked to the functions $\Psi^\pm(x)$ which describe the linear evolution of the system

$$\rho(x) = e^{2R(x)} = \Psi^+(x)\Psi^-(x).$$

Let us also remark that the Newton law (equation (4)) can be deduced from the conservation of the following energy (Nelson 1979)

$$E = \langle\tfrac{1}{2}m\{[\tfrac{1}{2}(D+D^*)x]^2-\lambda[\tfrac{1}{2}(D-D^*)x]^2\}+V(x)\rangle$$

$$= \int [\tfrac{1}{2}m(v^2(x)-\lambda\bar{u}^2(x))+V(x)]\rho(x)\,\mathrm{d}x.$$

If one further requires that this energy be positive definite ($\lambda < 0$), then, identifying the product of the diffusion parameter k and of the acceleration parameter $\sqrt{|\lambda|}$ with the universal Planck constant $\hbar$, one is led to the Schrödinger equation. The functions $\Psi^+(x)$ and $\Psi^-(x)$ are then conjugate of each other. On the other hand, the two other choices $\lambda > 0$, and $\lambda = 0$, lead to ordinary diffusion and to classical mechanics respectively. One then remarks that, to the same quantum state (the same wavefunction $\Psi^\pm(x)$, or else, the same fields $\rho(x)$, $u(x) = \sqrt{|\lambda|}\bar{u}(x)$, and $v(x)$), corresponds a whole family of different processes (different transitions probabilities, or else, different velocities $b(x)$, $b^*(x)$, and diffusion constant k). Hence, the size of the fluctuations remains arbitrary, the acceleration parameter $\sqrt{|\lambda|}$ matching in consequence. Nelson's convention corresponds to $|\lambda| = 1$, and thus $k = \hbar$.

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